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THE DISTRIBUTION OF THE POLES OF THE BEST APPROXIMATING RATIONAL FUNCTIONS AND THE ANALYTICAL PROPERTIES OF THE APPROXIMATED FUNCTION

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ABSTRACT

The following theorem was proved in our paper in Math. USSR-Sb. **9** (1969): If $f(z) \in H_2$ and the poles of the rational functions of best approximation tend to infinity sufficiently quickly, then $f(z)$ is an entire function. In the present article we weaken the restrictions on the distribution of poles by assuming only that these poles have no finite limit point (Theorem 1). Some generalizations of this result are also given.

1. Notation

$R_n^*(z;f)$—the best approximating rational function (below we consider in most cases the L_p-metric on $|z|=1$) for f in the class of all rational functions of degree not exceeding n, its poles being outside the unit disk $|z| \leqq 1$. We write simply $R_n^*(z)$ when it does not cause confusion.

$r_n(f) = \|f(z) - R_n^*(z;f)\|$.

$\|\cdot\|_1$ is the uniform norm on $|z|=1$.

$\|\cdot\|_T$ is the uniform norm on $|z|=T$.

$\|\cdot\|_{1,p}$ is the norm in L_p on $|z|=1$.

$\alpha_{n1}^*, \cdots, \alpha_{nn}^* = \{\alpha_{ni}^*\}_{i=1}^n$ are n poles of $R_n^*(z;f)$.

2.

Our main result is the following.

THEOREM 1. *Let $f(z)$ be an analytic function in the open disk $|z|<1$ and let $f \in H_p$. Let the set $\{\alpha_{ni}^*\}_{i=1}^{n} {}_{n=1}^{\infty}$ have no finite limit point. Then $f(z)$ is an entire function.*

REMARK. This theorem is proved in the case of L_p-approximation on $|z|=1$. It is readily verified from the proof that the analogous theorem holds in

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a uniform metric, if we strengthen the conditions of Theorem 1 by assuming that the degree of $R^*_n(z;f)$ is more than θn $(0<\theta<1$ and fixed).

Theorem 1 is the limit case of

THEOREM 2. *Let $f \in H_p$ and let the set $\{\alpha^*_{ni}\}_{i=1}^{n}{}_{n=1}^{\infty}$ have no limit point in the disk $|z|<A>3+2\sqrt{2}$.*† *Then $f(z)$ is analytic in the disk $|z|<\gamma(A)>1$, where $\gamma(A)=O(\sqrt{A})$ for large A.*

3.

In this section we prove Theorem 2.

LEMMA 1. *In $\phi(z)$ is analytic in the disk $|z| \leqq T>1$ and the points $\{\alpha_{ni}\}_{i=1}^{n}$ $(|\alpha_{ni}| \geqq A>1)$ are fixed, then there exists a rational function $R_n(z)$ with poles $\{\alpha_{ni}\}$ such that*

$$\|\phi - R_n\|_1 \leqq C\|\phi\|_T\left(\frac{A+T}{1+AT}\right)^n$$

where C depends on T, but not on ϕ, nor on n.

This lemma is another form of a result of Walsh [3, corol., p. 230]. The proof is simple, but it is included for completeness.

PROOF OF LEMMA 1. If $R_n(z)$ denotes the rational function of degree n whose poles lie in the points $\{\alpha_{ni}\}$, and which interpolates to $\phi(z)$ in the points $0, 1/\bar{\alpha}_1, \cdots, 1/\bar{\alpha}_n$, then we have (see [3, p. 186, formule 4])

$$\phi(z)-R_n(z)=\frac{1}{2\pi i}\int_{|t|=T}\frac{z}{t}\prod_{i=1}^{n}\frac{\bar{\alpha}_{ni}z-1}{z-\alpha_{ni}}\prod_{i=1}^{n}\frac{t-\alpha_{ni}}{\bar{\alpha}_{ni}t-1}\cdot\frac{\phi(t)}{t-z}\,dt$$

for $|z| \leqq 1$.

Since $|(\bar{\alpha}_{ni}z-1)/(z-\alpha_{ni})|=1$ for $|z|=1$ and since for $t=T$ holds $|(t-\alpha_{ni})/(\bar{\alpha}_{ni}t-1)| \leqq (A+T)/1+AT$ (see [3, p. 229, formule 8]), we have

$$|\phi(z)-R_n(z)|_{|z|=1} \leqq \left(\frac{A+T}{1+AT}\right)^n \|\phi(z)\|_T \cdot C,$$

where C obviously depends only on T. This completes the proof.

LEMMA 2. *If the poles of the rational function $R_n(z)$ lie outside the disk $|z|<A>1$, then for $1<T<A$ we have*

†This restriction is clarified in the sequel.

$$\| R_n(z) \|_T < C \| R_n(z) \|_{1,p} \left(\frac{AT-1}{A-T} \right)^n$$

where C depends on T but not on $R_n(z)$.

This Lemma is a special case of Lemma III in [3, p. 255].

LEMMA 3. *If* $f(z) \in H_p$ *and if for every n there exists a rational function* $R_n(z)$ *with poles* $\{\alpha_{ni}\}_{i=1}^n$ $(|\alpha_{ni}| \geqq A > 1$ *for all i, n) such that*

$$\overline{\lim_{n\to\infty}} \| f(z) - R_n(z) \|_{1,p}^{1/n} \leqq q < 1$$

then $f(z)$ *is analytic in the disk* $|z| < (A + q^{1/2})/(1 + Aq^{1/2})$.

This lemma (for a uniform metric) is identical with corollary 2 in [3, p. 233]. For the case of the L_p-metric it is readily proved by the same method using lemma I in [3, p. 231] for $r < 1$, and the lemma of §5.5 in [3, p. 101].

LEMMA 4. *Let* $f(z) \in H_p$ *and let* $f(z)$ *be not a rational function. Then in the case of approximation in the* L_p*-metric on* $|z| = 1$, *the degree of* $R_n^*(z; f)$ *is equal precisely to n.*

PROOF. By proposition 2 in [2] we have, under the conditions of Lemma 4, that $r_n(f) < r_{n-1}(f)$. This means that the degree of $R_n^*(z; f)$ is equal to n.

REMARKS

1. Only the fact that this lemma is false for a *uniform metric* prevents us from proving Theorem 1 for this case as well (see Remark in Section 3).
2. Referring to the proof of proposition 2 in [2] we can readily state that it is true for the L_p-approximation with a positive continuous norm function. Our Lemma 4 is thus valid for this case as well. We shall need this in Section 4.1.

LEMMA 5. *If the rational function* $R_n(z; f) = P_r^*(z)/Q_s^*(z)$ $(\max(r, s) = n)$ *is the rational function of best approximation for a given f in the norm* $\|\cdot\|$, *then*

$$\| f - R_n^* \| \leqq \left\| f - \frac{P_{2n}}{Q_s^{*2}} \right\|$$

for any polynomial P_{2n} *of degree* $2n$.

This lemma is a special case of theorem 4 in [1].

LEMMA 6. *Let* $\{r_n\}$ *be a non-increasing sequence of real numbers such that* $r_n \to 0$ *as* $n \to \infty$. *If* $\overline{\lim}_{n\to\infty} (r_n - r_{n+1})^{1/n} \leqq q < 1$, *then* $\overline{\lim}_{n\to\infty} r_n^{1/n} \leqq q$.

PROOF. Let $1 > q_1 > q$. Then if n is sufficiently large, we have by the assumption of Lemma 6 that $r_n - r_{n+1} \leqq q_1^n$.

Since $r_n \to 0$ we can write $r_n = \sum_{k=n}^{\infty}(r_k - r_{k+1})$. Estimating the last sum we have, for sufficiently large n, $r_n \leqq q_1^n/(1-q_1)$. So $\overline{\lim}_{n\to\infty} r_n^{1/n} \leqq q_1$. We can let q_1 approach q and this completes the proof.

PROOF OF THEOREM 2. Let R_n^*, R_{n+1}^* be the rational functions of best approximation (of degrees n and $n+1$ respectively) to a given f. By Lemma 4, the degree of R_n^* is exactly n. Set $R_n^*(z) = P_r^*(z)/Q_s^*(z)$ $(\max(r,s) = n)$. According to Lemma 5, we have for any polynomial P_{2n}

$$r_n(f) = \|f - R_n^*\|_{1,p} \leqq \left\| f - \frac{P_{2n}}{Q_s^{*2}} \right\|_{1,p} \leqq \|f - R_{n+1}^*\|_{1,p}$$

$$+ \left\| R_{n+1}^* - \frac{P_{2n}}{Q_s^{*2}} \right\|_{1,p}$$

$$= r_{n+1}(f) + \left\| R_{n+1}^* - \frac{P_{2n}}{Q_s^{*2}} \right\|_{1,p}.$$

There results the inequality

$$r_n - r_{n+1} \leqq \left\| R_{n+1}^* - \frac{P_{2n}}{Q_s^{*2}} \right\|_{1,p} \leqq \left\| R_{n+1}^* - \frac{P_{2n}}{Q_s^{*2}} \right\|_1. \tag{1}$$

We now proceed to estimate the right member in (1).

Let $1 < A' < A$. For sufficiently large n the poles of R_{n+1}^* and $P_{2n}(z)/Q_s^{*2}(z)$ lie outside the disk $|z| \leqq A'$. Hence, applying Lemma 1 for $\phi(z) = R_{n+1}^*(z)$, we can find a polynomial $P_{2n}^0(z)$ such that for any $1 < T < A'$ we have

$$\left\| R_{n+1}^* - \frac{P_{2n}^0}{Q_s^{*2}} \right\|_1 < C \| R_{n+1}^* \|_T \left(\frac{A' + T}{1 + AT} \right)^{2n}. \tag{2}$$

According to Lemma 2:

$$\| R_{n+1}^* \|_T \leqq C_1 \| R_{n+1}^* \|_{1,p} \left(\frac{A'T - 1}{A' - T} \right)^{n+1} \leqq 2C_1 \| f \|_{1,p} \left(\frac{A'T - 1}{A' - T} \right)^{n+1}. \tag{3}$$

Substituting (3) in (2), we obtain:

$$\left\| R_{n+1}^* - \frac{P_{2n}^0}{Q_s^{*2}} \right\|_1 \leqq C_2 \left(\frac{A'T - 1}{A' - T} \right)^{n+1} \left(\frac{A' + T}{1 + A'T} \right)^{2n} \tag{4}$$

Setting $P_{2n} = P_{2n}^0$ in (1), we obtain from (4):

$$(5) \qquad \overline{\lim_{n\to\infty}}\,(r_n - r_{n+1})^{1/n} \leqq \frac{A'T-1}{A'-T}\left(\frac{A'+T}{1+A'T}\right)^2 \text{ for any } 1<T<A'.$$

Letting A' approach A, we obtain from (5):

$$(6) \qquad \overline{\lim_{n\to\infty}}\,(r_n - r_{n+1})^{1/n} \leqq q, \text{ where } q = \min_{1<T<A}\left(\frac{AT-1}{A-T}\right)\left(\frac{A+T}{1+AT}\right)^2.$$

REMARK. The value of q is found without difficulty, although the expression is rather complicated. It should be noted that actually q is less than unity only if $A > 3+2\sqrt{2}$. This leads to the restriction on A in the statement of Theorem 2. For sufficiently large A $(A > 8)$, setting $T = A/3$ in (6) yields the simple estimate: $q < 8/A$. Now, provided $A > 3+2\sqrt{2}$, we can complete the proof.

By Lemma 6 we have from (6): $\overline{\lim}_{n\to\infty} r_n^{1/n} \leqq q$. From this inequality and Lemma 3 we may conclude that $f(z)$ is analytic in the disk

$$|z| < (A+q^{1/2})(1+Aq^{1/2}) = \gamma(A) > 1.$$

For sufficiently large A we have, recalling the above estimate of q, that $\gamma(A) = O(\sqrt{A})$. Theorem 2 (as well as Theorem 1) is thus established.

4. Extensions

4.1. Since an L_p-norm and L_p-norm with a positive continuous weight function are equivalent, we are able, using the same Lemmas 1–6, to prove Theorems 1, 2 for the case of the L_p-weight approximation. The proof is completely analogous.

4.2. The situation of Theorem 1 is invariant under linear transformation of the form $w = (\bar{\alpha}z - 1)/(z-\alpha)$, $|\alpha| > 1$, in the sense considered in [3, p. 226]: The best L_p-approximation (with a positive continuous norm function) to $f(z)$ on the unit circle by rational functions whose poles lie at the points α_{ni} corresponds to the best L_p approximation (with suitably chosen positive continuous norm function) on the unit circle in the w plane to the function which is the transform of $f(z)$ by rational functions whose poles lie at the transforms of the points α_{ni}. Thus we may state

THEOREM 1′. *Let $f \in H_p$ and let the set $\{\alpha^*_{ni}\}$ of poles of the best approximating rational function to $f(z)$ in the sense of least p-th powers on $|z| = 1$ with a positive continuous norm function have the unique limit point α ($|\alpha| > 1$). Then $f(z)$ is analytic in the entire plane except the point α.*

Theorem 2 can be generalized in a similar way.

4.3. Further generalization of Theorem 1′ can be obtained by replacing the condition whereby the set $\{\alpha^*_{ni}\}$ has the unique limit point, by the asymptotical condition (Walsh)

$$\overline{\lim_{n\to\infty}} \left| \prod_{i=1}^{n} \frac{\bar{\alpha}^*_{ni} z - 1}{z - \alpha^*_{ni}} \right|^{1/n} = |\psi(z)| \not\equiv \text{const.} \tag{7}$$

THEOREM 3. *Let $f(z) \in H_p$ and let relation (7) hold uniformly for z on an arbitrary closed subset of some region S. Let S contain the unit circle in its interior, but not the limit points of $1/\alpha^*_{ni}$. Further, let $S \cup \{|z| < 1\}$ contain the region R_T bounded by the locus $|\psi(z)| = T > 1$. Then $f(z)$ is analytic in $R_{T^{1/2}}$.*

The proof is analogous to that of Theorem 2, except for an appropriate minor modification of Lemmas 1–3. See for details [3, §9.4, theorem 7, lemma II and corol. 2].

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